Combining Galois with Complex Field Coding for Space-Time Communications

RENIU WANG  
Dept. of ECE, Univ. of Minnesota, Minneapolis, MN 55455, USA  
e-mail: renqiu@ece.umn.edu

ZHENGDAO WANG  
Dept. of ECE, Iowa State University, Ames, IA 50011, USA  
e-mail: zhengdao@iastate.edu

GEORGIOS B. GIANNAKIS  
Dept. of ECE, Univ. of Minnesota, Minneapolis, MN 55455, USA  
e-mail: georgios@ece.umn.edu

Abstract.  
A joint scheme combining error-control coding (ECC) with complex-field coding (CFC) is proposed for space-time communications through flat or frequency-selective fading multiple-input multiple-output (MIMO) channels. It is shown that the diversity gain of this joint coding scheme is the product of the free distance of the ECC, the complex-field block encoder size, and the number of receive antennas, which is computationally prohibitive to achieve using ECC alone. The decoding complexity with the proposed iterative decoder is comparable to a system without CFC. Numerical simulations for perfectly interleaved and HIPERLAN channels with both single and multiple antennas show great potential of the proposed joint coding scheme.

1 Introduction

Future generation communication systems are likely to deploy antenna arrays to enhance the data rates and cope with the adverse fading effects of wireless propagation. A number of space-time (ST) multi-antenna transmission schemes have been designed. Space-time orthogonal codes [16] offer performance-oriented designs: they can ensure maximum spatial diversity while their rate of transmission is at most one symbol per channel use. The V-BLAST scheme [23], on the other hand, is rate oriented, because it tries to maximize the transmission rate before performance is taken into account. There are also hybrid schemes that try to strike a desirable compromise between transmission rate and performance [11]. Elaboration on the trade-offs between transmission rate (or, the multiplexing gain) and performance in terms of diversity gain can be found in [27].

Linear constellation rotation coding [26] for space-time communications achieves a rate of one symbol per channel use with diversity gain as high as $MN$, where $M$ and $N$ are the number of transmit and receive antennas, respectively. Practical decoding, however, relies on near-optimum algorithms such as sphere-decoding [17] to guarantee maximum diversity. Since the complexity of sphere decoding increases dramatically with the number of independent symbols per block, in this case $M$, the application of constellation rotation codes is restricted to a small number of transmit antennas. The concept of constellation rotation, dates back to the signal-diversity concept for flat-fading channels [5], which was adapted to space-time applications in [6]. Recently, constellation rotation has also been applied to orthogonal frequency division multiplexing (OFDM) over frequency-selective fading channels, as a special case of the complex field (CF) coded OFDM [20]. Complex field coding (CFC) is the counterpart of a Galois-field (GF) block code: instead of having entries from a GF, a CFC generator can have complex entries. It has been shown in [20] and [26] that CFC allows for large (in many cases, maximum) diversity gain with small or no rate loss. The key point is that in fading channels, the minimum Hamming distance eventually determines the diversity order, or the slope, of the average probability of error curve. Even a square (that is, rate 1) CFC is capable of producing codewords that have minimum Hamming distance as large as the codeword length, which is rarely possible with GF codes.
We propose in this paper a concatenated coding scheme for space-time communications. The idea is to combine conventional error control coding (ECC) and CFC in a serial fashion, with an interleaver in between. Serial concatenation of two ECCs has been used for a long time, where the outer code is usually a Reed-Solomon code, while the inner code can be either a block code or more often a convolutional code. Recently, serially concatenated convolutional codes (SCCC) [2] have become popular as an alternative form of the powerful parallel concatenated convolutional codes (PCCC), known otherwise as turbo codes [4]. The concatenation of ECC and CFC, however, is motivated by the observation that ECC is usually very good in coping with additive noise channels, while CFC, thanks to its large Hamming distance, is quite effective in coping with fading channels. Transmitting CFC encoded symbols over the channel and detecting them at the receiver convert the fading channel to a “less-fading” one, ideally to an additive white Gaussian noise (AWGN) channel when the CFC encoder size grows to infinity, for which the ECC is known to become most effective [4].

The combination of ECC with CFC has been used in [22] to cope with frequency-selectivity, and has been advocated in [13] for OFDM systems, where the CFC takes the special form of a Hadamard matrix and the scheme is referred to as code-division multiplexed OFDM. In both cases, improved performance has been observed. With pairwise error probability (PEP) analysis, it has been shown in [22] that the diversity gain with a perfect interleaver is actually the free distance of the ECC (if it is a convolutional code), times the CFC encoder size. This multiplicative diversity effect will also be true in our joint ECC-CFC space-time transmission scheme, except that the diversity gain will pick an additional multiplicative factor \(N\), the number of receive antennas.

We will present the system model in section 2, where using OFDM, we will convert an infinite-impulse response (FIR) multiple input multiple-output (MIMO) channel to a set of correlated flat fading MIMO channels. We will then introduce the idea and some important results pertaining to CFC for fading channels in section 3. We will develop the joint ECC and CFC scheme in section 4, including the performance analysis, and a sketch of a low complexity iterative decoding algorithm. We will finally report simulation results in section 5, and conclude the paper in section 6.

Notation: we will use bold face lowercase (uppercase) letters to denote column vectors (matrices); \((\cdot)^T\) and \((\cdot)^H\) will denote transpose and hermitian transpose respectively; \(\text{diag}(x_1, \ldots, x_n)\) is an \(n \times n\) diagonal matrix with the given entries on the diagonal; \(|\cdot|\) is the Euclidean norm; and \([X]_{i,j}\) is the \((i, j)\)th entry of the matrix \(X\).

2 Space-time MIMO channel

We consider space-time multiple-antenna transmissions over a frequency-selective channel. Let \(M\) denote the number of transmit antennas, and \(N\) the number of receive antennas. Accounting for transmitter pulse shaping, receiver filtering, and symbol period sampling, the continuous channels are converted to discrete-time baseband equivalent channels. The equivalent channel from the \(m\)th transmit-antenna to the \(n\)th receive antenna will be denoted as \(h_{n,m}(l)\), where \(m \in [1, M]\), and \(n \in [1, N]\). We assume that the channel is known at the receiver and the receive filter is matched to the channel and the transmitter pulse, so that symbol rate sampling produces sufficient statistics for symbol decision and hence does not lead to information loss (see, e.g., [9]). When the channel is unknown at the receiver, higher than Nyquist rate sampling should be used to avoid information loss in the sampling process. In addition, we assume that all \(MN\) channels are FIR, and the maximum order is upper bounded by \(L\). In practice, \(L\) is estimated by the dividing maximum delay spread with the sampling period.

2.1 Flat fading MIMO channel

When \(L = 0\), the MIMO channel is flat-fading, and the input-output relationship can be written as

\[
y = Hx + v, \tag{1}
\]

where \(x, y, v\) are \(M \times 1\), \(N \times 1\), and \(N \times 1\) vectors denoting transmitted signal, received signal, and additive noise samples across the multiple antennas during one symbol period, respectively; \(H\) is the channel mixing matrix, whose \((n, m)\)th entry is \(h_{n,m}(0)\), or, for simplicity, \(h_{n,m}\).

We next confirm that by using OFDM, our MIMO FIR channel with \(L > 0\) reduces to a set of flat-fading MIMO channels like (1) with no intersymbol interference (ISI). This will facilitate the equalization and decoding at the receiver, and will also simplify the description of our system model.

2.2 ISI-free MIMO channels using OFDM

Consider the block diagram in figure 1, where \(x_m := [x_m^{(1)}, x_m^{(2)}, \ldots, x_m^{(K)}]^T, m = 1, \ldots, M,\) are blocks of symbols to be transmitted, and \(K\) is the number of subcarriers in the OFDM system. We will index the transmit antenna by \(m\), and receive antenna by \(n\), both in subscript. The superscript in parenthesis will index the subcarriers. We assume that the MIMO channel remains approximately invariant over \(K + L\) symbols periods.

For each \(m\), the block \(x_m\) is first processed using a \(K\)-point Inverse Fast Fourier Transform (IFFT) and then a cyclic prefix of length \(L\) is inserted before each length-\(K\) IFFT output vector. The cyclic prefixed block is
parallel-to-serial converted, and transmitted through the
mth FIR channel (i.e., the mth transmit antenna) over
K + L time slots. At each receive antenna, every K + L
samples are collected, the first L of which correspond
to the cyclic prefix, and are removed because they con-
tain interference from the previous block. The remain-
ung K samples are serial-to-parallel converted, and pro-
cessed using FFT. The resulting K × 1 vector is denoted
by \( y_n := [y_n^{(1)}, y_n^{(2)}, \ldots, y_n^{(K)}]^T \) for the nth antenna. When there is only one transmit antenna, say the mth transmitting, \( y_n \) can be shown as in the single-input single-output (SISO) channel case to be (see, e.g., [19])

\[
y_n = D_{n,m} x_m + v_n, \tag{2}
\]

where \( D_{n,m} \) is a \( K \times K \) diagonal matrix whose \((k,k)\)th entry is

\[
H_{n,m}(k) := \sum_{l=0}^{L} h_{n,m}(l) \exp(-j2\pi kl/K),
\]

i.e., the frequency response of \( h_{n,m}(l) \) at frequency \( k/K \); and \( v_n := [v_n^{(1)}, v_n^{(2)}, \ldots, v_n^{(K)}]^T \) is the FFT processed AWGN at the nth receive antenna.

Using the linearity of the system, when all the anten-
as are transmitting, we can write the receive vectors \( y_n \)
together as follows

\[
\begin{bmatrix}
  y_1 \\
  \vdots \\
  y_N
\end{bmatrix}
= 
\begin{bmatrix}
  D_{1,1} & \cdots & D_{1,M} \\
  \vdots & \ddots & \vdots \\
  D_{N,1} & \cdots & D_{N,M}
\end{bmatrix}
\begin{bmatrix}
  x_1 \\
  \vdots \\
  x_M
\end{bmatrix}
+ 
\begin{bmatrix}
  v_1 \\
  \vdots \\
  v_N
\end{bmatrix}. \tag{3}
\]

It is clear now that the FIR MIMO channel with M in-
puts and N outputs has been converted into a larger mem-
oryless MIMO channel with \( MK \) inputs and \( NK \) outputs.
The \( NK \times MK \) channel mixing matrix, however, has special structure: it is sparse with diagonal block entries.

Instead of arranging the data according to the anten-
a indices, we can also arrange them according to subcarrier indices. Specifically, if we define the vectors \( x^{(k)} := [x_1^{(k)}, x_2^{(k)}, \ldots, x_M^{(k)}]^T \), \( y^{(k)} := [y_1^{(k)}, y_2^{(k)}, \ldots, y_N^{(k)}]^T \), \( v^{(k)} := [v_1^{(k)}, v_2^{(k)}, \ldots, v_N^{(k)}]^T \), and the \( N \times M \) mixing matrices \( H^{(k)} \), whose \((n,m)\)th entry is \( H_{n,m}(k) \), we

\[
\begin{bmatrix}
  y^{(1)} \\
  \vdots \\
  y^{(K)}
\end{bmatrix}
= 
\begin{bmatrix}
  H^{(1)} & \cdots & 0 \\
  \vdots & \ddots & \vdots \\
  0 & \cdots & H^{(K)}
\end{bmatrix}
\begin{bmatrix}
  x^{(1)} \\
  \vdots \\
  x^{(K)}
\end{bmatrix}
+ 
\begin{bmatrix}
  v^{(1)} \\
  \vdots \\
  v^{(K)}
\end{bmatrix}, \tag{4}
\]

where the mixing matrix is now block diagonal. Since the AWGN vectors \( \{v^{(k)}\}_{k=1}^{K} \) are independent, Equation (4) shows that using OFDM, we have converted the FIR MIMO channel into \( K \) memoryless MIMO channels:

\[
y^{(k)} = H^{(k)} x^{(k)} + v^{(k)}, \quad k = 1, 2, \ldots, K. \tag{5}
\]

When the FIR channels \( h_{n,m}(l), m \in [1, M], n \in [1, N] \) are treated as deterministic, the \( K \) MIMO channels in (5) are independent: they have independent additive noise blocks, and are capable of transmitting independent data blocks. When the channels \( h_{n,m}(l), m \in [1, M], n \in [1, N] \) are treated as random, the mixing matrices of the \( K \) MIMO channels in (5) are also random. They are usually correlated and therefore not statistically independent, although they are also capable of transmitting independent data streams, and the additive noise blocks are independent.

For our purpose, we will use (5) as our base model. We assume that all the FIR taps \( h_{n,m}(l) \) are circular complex Gaussian distributed, and hence they have Rayleigh distributed amplitudes, and uniformly distributed phases that are independent of the amplitudes. All entries of \( H^{(k)} \) will then also be circular complex Gaussian distributed. We can therefore view each of the \( K \) individual MIMO channels in (5) as an MIMO channel corresponding to a space-time communication system with \( M \) transmit and \( N \) receive antennas, over flat-fading channels [cf. (1)]. If all the \( MN \) FIR channels are statistically independent, then each of the \( K \) MIMO channels in (5) will have statistically independent entries. But in general, the entries of two different equivalent MIMO channels, say \( H^{(k)} \) and \( H^{(k')}, k \neq k' \) in (5), will be correlated.

With this observation, we will assume in the follow-
ing, without loss of generality, that the original MIMO is
flat fading; that is \( L = 0 \). The flat fading MIMO channel
will be correlated in consecutive usages of it, either due to
the frequency-domain cross-subcarrier correlation as in

![Figure 1: ISI-Free Parallel MIMO Channels Using OFDM](image-url)
A linear combination can be recovered from the channel. The encoding is performed simply through the matrix-vector product:

\[ u = \Theta s \]

where \( \Theta \) is the coding matrix \( \Theta \) is square, it does not have any common entry. With this property, the entire vector \( s \) can be recovered even from one entry of \( u \). We call a CFC that satisfies this condition maximum Hamming distance separable (MHDS).

Suppose for a moment that the channel is independently and identically distributed (i.i.d.), flat fading Rayleigh, and SISO; see figure 2. The entries of \( u \) will be transmitted serially through the channel. Denoting the channel corresponding to the \( P \) entries of \( u \) as a diagonal matrix \( A = \text{diag}(a_1, \ldots, a_P) \), where \( a_1, \ldots, a_P \) are the \( P \) i.i.d. channel attenuation coefficients, the received vector \( r \) can be written as

\[ r = A\Theta s + v = \text{diag}(\theta_1^T s, \ldots, \theta_P^T s) a + v, \tag{6} \]

where \( \theta_p^T \) is the \( p \)th row of \( \Theta \), \( p = 1, \ldots, P \), \( a := [\alpha_1, \ldots, \alpha_P]^T \), and \( v := [v_1, \ldots, v_P]^T \) is the additive noise.

The average performance of detecting \( s \) given \( r, a, \) and \( \Theta \) can be derived using PEP analysis. An error vector \( e = s - s' \), for \( s \neq s' \) will result in a received error vector

\[ r_e := \text{diag}(\theta_1^T e, \ldots, \theta_P^T e) a. \tag{7} \]

The average PEP for i.i.d. Gaussian distributed \( a \) is quantified by two parameters: the diversity gain \( G_d \), and the coding gain \( G_c \). The diversity gain is equal to the minimum rank of the matrix \( \text{diag}(\theta_1^T e, \ldots, \theta_P^T e) \) over all nonzero \( e \), and when this matrix is full rank, the coding gain \( G_c \) is quantified by the minimum determinant of it for all nonzero \( e \)'s. By the design of \( \Theta, \text{diag}(\theta_1^T e, \ldots, \theta_P^T e) \) is always full rank for any nonzero \( e \), and therefore full diversity \( P \) is achieved [24–26].

### 3.1 CFC for ST: rate one case

This signal space diversity has been exploited recently with multi-antenna transmissions in [6, 25]. The coding gain (or, the determinant) has been optimized in [25] using algebraic number theory. We adopt here the scheme proposed in [25] for our joint space-time ECC-CFC coding setup.

Specifically, we transmit the entries of the CF encoded vector \( u \) serially using the antennas, one entry per symbol period. When there is only one receive antenna, the received signal will be coming from only one transmit antenna at one time (no two antennas are transmitting together). Since the CF coding matrix \( \Theta \) is square, it does not introduce rate loss: the rate of transmission is one symbol per channel use. It may seem that the multiple antennas are not used efficiently: one could also transmit using the multiple antennas simultaneously, and thus increase transmission rate. Some high rate options that allow for such parallel transmissions will be presented in section 3.2. But even for rate one transmissions, antenna switching essentially increases the equivalent channel variation speed by sampling the different channels periodically, and alternately.

There are other ways of converting the multiple-input channel to a single-input channel. Orthogonally designed space-time codes offer one such scheme [16]. However, for complex constellations, a rate one orthogonal design is achievable only for two transmit antennas. For more than two transmit antennas, there is a rate loss factor of 3/4 or 1/2 [16]. Other means of converting the multiple-input channel to a single-input channel include antenna phase-sweeping [12], and delay diversity transmissions [14]. We will compare our CFC antenna switching scheme with orthogonal designs using simulations in section 5. We will also try to combine CFC with orthogonal designs for the two transmit-antenna case. When delay diversity is used, a flat fading ST channel is converted into a frequency-selective channel, and existing results pertaining to OFDM systems are directly applicable.

When considering antennas transmitting alternately through a flat-fading MIMO channel, the MIMO channel is essentially converted into a single-input multiple-output (SIMO) channel; see figure 3. At the receiver, channel state information (CSI) is assumed available, and maximum ratio combining (MRC) is performed on the received signal. The equivalent SIMO channel is then converted into...
Nakagami-m channel with parameter m. The probability distribution function (PDF) of the squared sum of the same SISO flat-fading channel, whose SNR is a parallel SISO channels. Assuming the entries of X are i.i.d., the resulting MIMO channel has been converted into a diagonal channel, which can be viewed as M parallel SISO channels. Assuming the entries of H are i.i.d., the resulting M SISO channels are also independent. The signal-to-noise ratio (SNR) at the mth subchannel output is given by \( \sum_{n=1}^{N} |h_{n,m}|^2 \); all M subchannels have the same statistics, and they can therefore be viewed as M SISO flat-fading channel, whose SNR is a squared sum of N Rayleigh distributed random variables. The probability distribution function (PDF) of the equivalent SISO channel is therefore a chi-square function with \( 2N \) degrees of freedom, in which case the channel is also called Nakagami-m channel with parameter \( m = N \) [15].

Now, the CF coded symbols will be sent through the equivalent SISO channel. That is, the entries of \( \hat{x} \) will be CF coded symbols. The received signal corresponding to one CF coded block can again be described by (6), except that the channel matrix A now has Nakagami distributed diagonal entries. Supposing that s contains uncoded symbols, the error probability performance can be evaluated using PEP analysis by considering the received error vector of (7), with a having Nakagami distributed entries. The PEP for one realization of a is given by \( Q(\sqrt{|r_c|^2/(2N_0)}) \), where \( N_0/2 \) is the noise power spectrum density, and

\[
|r_c|^2 = \sum_{p=1}^{P} |\theta_p^\dagger e|^2 |\alpha_p|^2.
\]

Averaging (11) with respect to the \( \alpha_p \) Nakagami-m distributed random variables will yield the average PEP. We will use the following two lemmas we have proved in [21].

**Lemma 1** (Diversity for Nakagami-m Channel)
Each Nakagami-m distributed random variable can offer diversity m. In our case, the m parameters of \( \alpha_p \)'s are all equal to N.

**Lemma 2** (Sum Diversity Rule)
An SNR that is a linear combination of the form of \( \sum_{b=1}^{B} c_b |\alpha_b|^2 \) for positive constants \( c_b \) and independent \( \alpha_b \)'s offers a diversity order that is equal to the sum of diversity orders achieved by each \( \alpha_b \).

When the CFC is MHDS, all the combining coefficients \( |\theta_p^\dagger e|^2 \) in (11) are non-zero. It follows that the total diversity achieved by CFC is \( \sum_{p=1}^{P} N = PN \). Notice that this is achieved under the assumption that \( \{\alpha_p\}_{p=1}^{P} \) are independent. This is usually true when \( P \leq M \). When \( P > M \), transmission of one CF coded block will involve the same antenna at least twice. In this case, interleaving can be used across time, or frequency in the frequency-selective case, to decorrelate the channel.

We have not touched upon the decoding options for CFC. For more information, the reader is referred to [20]. However, we will only be using encoders of small sizes (\( P = 2 \), or \( P = 4 \)), in which case, even exhaustive enumeration is computationally feasible.
3.2 CFC for ST: high rate cases

So far, we have discussed CFC with a transmission rate of one uncoded symbol per channel use. Although the rate is already higher than most GF coding schemes with rate less than one, it is possible to achieve even higher rate transmissions using CFC. We outline a few possibilities in the following. They will not be fully elaborated, however, due to lack of space.

**Compressive CFC.** Instead of a square CFC matrix $\Theta$, we can also use a $\Theta$ of size $P \times P'$ that has more columns than rows; i.e., $P < P'$. The CFC encoded block is again serially transmitted through the equivalent SISO channel in figure 3. Such a $\Theta$ can be obtained by truncating a larger CFC $\Theta'$ of size $P' \times P'$, and keeping only $P < P'$ rows of $\Theta'$. If $\Theta'$ is designed to achieve full diversity so that any two blocks encoded by $\Theta'$ do not share even a single entry, then $\Theta$ will also inherit full diversity because a block encoded by $\Theta$ is only a sub-block of one encoded by $\Theta'$. The resulting system will then have diversity order $PN$, the same as in the rate one case.

**Triangular ST signaling.** Another possible scheme that can improve transmission rate is to allow for simultaneous (as opposed to alternate) transmissions from the multiple transmit-antennas. In order to guarantee maximum diversity, we propose to transmit using a triangular ST signal matrix. Specifically, consider the model (8). Instead of having a diagonal matrix $X$, which corresponds to antenna-switching, we let $X$ be lower (or upper) triangular. The entries of $X$ are GF coded symbols that are outputs of a size $M(M+1)/2$ CFC encoder: there are totally $M(M+1)/2$ non-zero entries in the matrix $X$. The rate in this case is $(M+1)/2$ symbols per channel use. If the CFC is MHDS, it can be shown that such a signaling scheme can guarantee maximum diversity order of $MN$. The proof, however, is omitted due to lack of space.

In the following, we will focus our analysis on the rate one case of section 3.1, and will only examine the aforementioned higher rate options using simulations.

4 Joint Galois field and complex field coding

The complexity of optimum CFC decoding increases exponentially with the size parameter $P$. Choosing $P$ presents a tradeoff between higher diversity, and lower complexity. The same tradeoff also emerges with GF codes (GFC). For example, consider a convolutionally coded transmission through an i.i.d. flat-fading Rayleigh channel. The diversity order is the free distance $d_{\text{free}}$ of the GF code. For a fixed rate code, the free distance can be increased by increasing the encoder memory. This memory increase, however, results in an exponential increase in decoding complexity, because the number of states in the trellis of the code is exponential in the encoder memory.

A better tradeoff between performance and complexity is offered when combining GFC with CFC in a serial fashion. We have introduced this combination for single-antenna OFDM transmissions in [22], where the performance of such a joint coding system has been studied using PEP analysis under the assumption of perfect interleaving. In the following, we will tailor the major results of [22], to our space-time setup.

4.1 Serial concatenation of ECC with CFC

Consider the system diagram in figure 3. At the transmitter, the information bit stream $b_n$ is first encoded using some GF error control mechanism involving e.g., block, convolutional, or turbo codes. The output stream $c_n$ is then interleaved using $\Pi_1$, and mapped to constellation symbols. If Trellis Coded Modulation (TCM) is used as the error-control code, then the constellation mapping is not needed since the mapping is already incorporated in the TCM.

After constellation mapping, the information-bearing symbol stream is encoded by a CFC matrix $\Theta$. For low-complexity decoding, only small matrices of size 2 or 4 will be used. Specifically, we will use the following two encoders designed using algebraic number theory [22, 26]

$$\Theta_2 = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & e^{j\frac{\pi}{4}} \\ 1 & e^{j\frac{3\pi}{4}} \end{bmatrix}, \quad \Theta_4 = \frac{1}{2} \begin{bmatrix} 1 & e^{j\frac{\pi}{8}} & e^{j\frac{3\pi}{8}} & e^{j\frac{5\pi}{8}} \\ e^{j\frac{\pi}{8}} & e^{j\frac{3\pi}{8}} & e^{j\frac{5\pi}{8}} & e^{j\frac{7\pi}{8}} \\ e^{j\frac{9\pi}{8}} & e^{j\frac{13\pi}{8}} & e^{j\frac{17\pi}{8}} & e^{j\frac{21\pi}{8}} \\ e^{j\frac{13\pi}{8}} & e^{j\frac{17\pi}{8}} & e^{j\frac{21\pi}{8}} & e^{j\frac{25\pi}{8}} \end{bmatrix}. \tag{12}$$

For comparison purposes, we will also use Hadamard matrices, that we denote as $\Theta_{hd}$:

$$\Theta_{hd}^2 = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}, \quad \Theta_{hd}^4 = \frac{1}{2} \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 \end{bmatrix}. \tag{13}$$

Such Hadamard matrices have been advocated in [13], where code-division multiplexing is combined with OFDM to exploit frequency domain diversity.

After the CF block encoding by $\Theta$, the symbol stream $u_n$ is interleaved by $\Pi_2$, and the interleaver output $\tilde{u}_n$ is transmitted over the multiple antennas alternately, using one transmit-antenna per time slot. When the channel is frequency-selective, we assume that OFDM modulation/demodulation has been used to convert the channel into parallel (correlated) flat fading channels, as in (5).

The choice of the size of $\Pi_2$ depends on the channel variation. For slowly varying channels, $\Pi_2$ needs to have a large enough size so as to sufficiently decorrelate the channel. For performance analysis, we will assume that the channel has been perfectly decorrelated so that the
equivalent SISO channel in figure 3 is i.i.d., Nakagami-m distributed with parameter $m= N$, the number of receive-antennas.

Let $R$ denote the information rate of the ECC, and $Q$ denote the size of the constellation used before CFC. The rate of the overall system is therefore $KR\log_2(Q)$ bits per channel use. When the triangular signaling scheme of section 3.2 is used, the rate can be $(M+1)/2$ times higher, reaching $0.5KR(M+1)\log_2(Q)$ bits per channel use.

### 4.2 Performance analysis

We assume that the ECC is a convolutional code properly terminated. Other choices including block or turbo codes can be treated similarly. We next look into performance using PEP analysis. For simplicity, we assume binary phase-shift keying (BPSK), although other constellations can be dealt with likewise.

Consider a pair of different information sequences that are encoded using ECC. They may result in two ECC codewords that are different in $w$ ECC symbols, where $w \geq d_{\text{free}}$, and $d_{\text{free}}$ is the free distance of the ECC. After interleaving, these $w$ non-zero symbols in the error sequence will be spread far away from each other (or at least so in a statistical sense when a random interleaver $\Pi_1$ is used). The CFC is then used to spread each of these $w$ symbols to $P$ independent flat-fading channel attenuations. The remaining symbols in one CFC block are all zeros, under the assumption that no two of the $w$ non-zero symbols in the ECC error event enter the same CFC block. Each of the $wp$ non-zero symbols at the CFC output will have an amplitude $\sqrt{E_s/P}$, where $E_s$ is the energy per ECC symbol, and the factor $\sqrt{P}$ comes from the energy normalization factor of each CFC matrix (see (12) and (13)). Each of the $wp$ symbols sees a different realization of the equivalent SISO channel in figure 3. The resulting received error sequence will thus have $wp$ non-zero symbols:

$$\bar{y}_i = \alpha_i u_i, \quad i = i(j), j = 1, 2, \ldots, wp,$$

where $u_i$ is the $i$th non-zero symbol in the CFC output of the error event; $\alpha_i$ is the flat SISO symbol that $u_i$ sees, which is the square root of the squared sum of $N$ Rayleigh random variables that gives $E[\alpha_i^2] = N$ [cf. (10)]; and $y_i$ is the corresponding received signal from the equivalent SISO channel of figure 3. The additive noise term that can confuse the decision between the two sequences has variance $N_0$. The probability of a pairwise error between the pair of information sequences is then given by the function

$$Q(\sqrt{\sum_{i=1}^{wp} |y_i|^2/(2N_0)}) = Q(\sqrt{\sum_{i=1}^{wp} |\alpha_i u_i|^2/(2N_0)}).$$

(15)

Using the two lemmas of section 3.1, it is easy to see that the diversity order for the pairwise error is $wPN$. Another way to see this is to write each $|\alpha_i|^2$ as the squared sum of $N$ independent Rayleigh random variables [cf. (10)]. Since $|u_i|^2 = E_s/P$, each $|y_i|^2$ term in (15) will be equivalent to a squared sum of $N$ Rayleigh random variables each having second moment equal to $E_s/P$. The variable $\sum_{i=1}^{wp} |\alpha_i u_i|^2/(2N_0)$ will then be chi-square distributed with $2wpN$ degrees of freedom, giving rise to a diversity order of $wPN$.

The minimum of this diversity order among all error events is $d_{\text{free}}P$, and will dominate the overall system performance. From this, we conclude that the system diversity order is $d_{\text{free}}P$. We call this the multiplicative diversity effect. That is the system diversity is the product of $d_{\text{free}}$, the diversity achieved by the convolutional code when used alone, times $P$, the diversity achieved by the CFC when used alone, times the receive diversity $N$. The number of transmit antennas, $M$, does not play an apparent role here, because we have assumed that antenna switching is used, and the equivalent SISO channel has been sufficiently decorrelated. However, $M$ will show up if we consider the performance of the high-rate triangular signaling scheme of section 3.2.

Although the assumption that all non-zero symbols in the ECC error event enter different CFC blocks is not always true, it is quite valid especially for small weight error events. For large weight error events, even when two more non-zero symbols enter the same CFC encoding block, the effect on the error rate will not be large, simply because such large weight events have small error probability.

The coding gain and the gain of using CFC have also been quantified in the single-antenna OFDM setup in [22]. We will not re-derive them here, but only mention that using CFC after ECC results in a gain that increases with the CFC encoder size. And with size 4 CFC encoders, most (70-80%) of the ultimate gain achievable by using an infinitely large CFC can already been achieved. Thus, from a performance-complexity tradeoff point of view, using a CFC encoder of size larger than 4 is not recommended.

### 4.3 Low complexity iterative decoding

At the receiver, we use iterative decoding whereby information is exchanged between two soft-input soft-output (siso) modules that we abbreviate as siso-CF and siso-GF; see figure 4. The siso-CF module is the maximum a posteriori (MAP) decoder for the CF block code $\Theta$, and produces soft information about the GF-coded symbols. The siso-GF module implements the MAP decoder for the error-control code $[3]$.

The siso-CF module accepts two inputs: the log-likelihood ratio of the interleaved bits $d$ (cf. figure 3), and the log-likelihood of the CFC block (or, CFC codeword); the latter is obtained from the received signal, while the former is obtained from the siso-GF block. The siso-CF module produces two outputs: the extrinsic information of the CFC codeword, which is not used by siso-CF, and
Figure 4: Iterative decoding algorithm

the extrinsic information of the interleaved ECC bits \( d \). The extrinsic information is defined as the a posteriori log-likelihood ratio minus the a priori log-likelihood ratio. For more information on the extrinsic information, see [3]. In order to compute the a posteriori log-likelihood ratio in our case, we simply enumerate all the \( 2^P \) combinations of all bits that are encoded into one CFC block [22]. The complexity of this step is small for \( P = 2 \) and \( P = 4 \). When a large \( P \) is used, a soft interference cancellation scheme can be used; see, e.g., [18].

The extrinsic information of \( d \) is de-interleaved and fed into the siso-GF module, which also accepts the uniform distribution (or, zero log-likelihood ratio) of the information bits and computes the extrinsic information of the information bits and the ECC bits \( c \). The latter is interleaved and used as the a priori information for the siso-CF module. The computation involved in their siso-GF module is pretty standard, see e.g., [3].

The complexity of the decoding algorithm depends on the individual complexities of the siso-CF and siso-GF modules, and the number of iterations. In our simulations, we have observed that only two iterations suffice in most cases: more iterations do not offer significant additional gains. The siso-GF module complexity is the about same as the decoding complexity of an ECC system without CFC. The complexity of siso-CF is small when \( P = 2 \) or \( P = 4 \), and is therefore not dominant when compared to the siso-GF module. In summary, the total complexity of decoding the joint ECC-CFC system is only about two times that of an ECC-only system (assuming two iterations). We have also observed that even with one iteration, there is already a gain as compared to an ECC-only system.

5 Simulation results

In this section, we show numerical simulation results of our joint ECC-CFC scheme for different setups. We use BPSK constellation, and \( E_b/N_0 \) to denote the bit SNR per receive antenna. We implement the CFC encoders (12) except in Test Case 2. Unless otherwise mentioned, we adopt throughout a random interleaver \( \Pi_1 \) of length corresponding to a delay of 1024 information bits, which we call a frame of symbols. Interleaver \( \Pi_2 \) is assumed to be perfect so that the channel is completely decorrelated in time, except for the HIPERLAN simulation in Test Case 4. We stop the bit-error rate (BER) count when either 50 frame errors are detected, or, 2000 frames are transmitted. Except for Test Case 1, we use one receive antenna: \( N = 1 \).

Test case 1 (Multiplicative diversity effect): In this simulation, we assume that the channel is decorrelated across time and antennas. We test the antenna-switching scheme of figure 3, where the MIMO channel is converted to a flat-fading i.i.d., SISO channel; cf. (10). We use two iterations for decoding the joint ECC-CFC system. More iterations did not provide additional gains. Two iterations were used also for ECC-CFC. The ECC was a rate 1/2 convolutional code with constraint length 3, generating polynomials in octal form (7, 5), and \( d_{free} = 5 \).

We tested the ECC without CFC, and ECC with CFC of size \( P = 2 \), both for one receive antenna (\( N = 1 \)), as well as ECC with CFC of size \( P = 2 \) for two received antennas. From the BER performance in figure 5, it can be seen that there is a performance improvement by using CFC. Even one iteration provides quite noticeable gain, while the diversity orders approximately comply with the multiplicative diversity prediction: the ECC-only system has diversity equal to \( d_{free} = 5 \), the ECC-CFC with \( N = 1 \) and
Combining Galois with Complex Field Coding for Space-Time Communications

$P = 2$ has diversity 10, and the ECC-CFC with $N = 2$ and $P = 2$ has diversity order close to 20. The huge gain in the $N = 2$ case is also due to the additional receive antenna, which increases the total received power by 3dB.

**Test case 2 (Hadamard CFC):** In this simulation, we tested the effect of different CFC choices on performance. With BPSK constellation, we compare ECC-CFC with $N = 1$ and $P = 2$ for perfectly interleaved channels. The ECC is the same as the one in Test Case 1. The encoder $\Theta_2$ of (12), and $\Theta_{hd}^b$ of (13), which was not optimized for its coding gain, were compared. The simulation results with 2 iterations are depicted in figure 6, where we can see that both CFC encoders offer a boost of diversity. But the CFC encoder $\Theta_2$ offers a better coding gain in SNR as compared to the Hadamard encoder. It can be easily verified that the Hadamard encoder is not MHDS for the BPSK signal set, but because of the interleaver $\Pi_1$, between the ECC and CFC, the diversity gain is still multiplicative because the non-MHDS property of the Hadamard CFC only affects ECC error events of large weight.

**Test case 3 (Comparison and combination with Alamouti’s code):** Relying on two transmit antennas, we compare here our ECC-CFC antenna switching scheme with the Alamouti code [1], which also has rate one symbol per channel use. Both schemes were encoded using a rate 1/2 convolutional code with generators $(7, 5)$. We depict the performance of our ECC-CFC scheme with two iterations along with that of the ECC encoded Alamouti code in figure 7. It can be seen that there is virtually no difference in the BER.

We can also consider using CFC and Alamouti coding jointly, in which case, we will replace our antenna-switching strategy with the Alamouti code. The result is also depicted in figure 7. Although the additional diversity gain shows up at high SNR, combining the two only offers coding gain of about 0.5dB for low SNRs.

**Test case 4 (HIPERLAN Simulations):** We simulated in this test the proposed scheme in a HIPERLAN setup using the HIPERLAN channel model A [7, 8], with a carrier frequency 5.2 GHz and speed 3 m/s. We used the equivalent correlated flat fading MIMO model that we derived in section 2.1. $\Pi_1$ was chosen to be a random interleaver with a delay corresponding to 16,384 information bits, while $\Pi_2$ was chosen to be a $4 \times 16$ block interleaver. Transmit antenna switching was used. We depict the results in figure 8 with $M = 1$ and $M = 4$, with or without CFC. The ECC was a rate $3/4$ convolutional code punctured from the rate $1/2$ (171, 133) code used in the HIPERLAN standard [8]. We can see that with CFC, we obtain at BER $= 10^{-3}$ a gain in SNR of 2 dB and 3 dB for $M = 1$ and $M = 4$, respectively, as compared to the ECC-only case without CFC.

**Test case 5 (High rate triangular signaling):** We simulated in this test the performance of the high rate triangular signaling scheme of section 3.2. We chose $M = 3$ and $N = 1$. According to section 3.2, the transmission rate is $(M + 1)/2 = 2$ symbols per channel use. Using the rate 1/2 $(7, 5)$ convolutional code, the information rate was 1 bit
per channel use. At the receiver, we again used siso modules for iterative decoding. We depict in figure 9 the BER performance of the ECC-CFC coded transmission with antenna switching and with 3 decoder iterations, together with the performance of an ECC-only system with antenna switching but without CFC high-rate encoding. Although the rate of the ECC-only system is only 0.5 bit per channel use, half of that of the ECC-CFC system, the ECC-CFC has better performance for $E_b/N_0 > 2\, \text{dB}$. The price paid is slightly increased complexity.

6 Conclusions

We have developed in this paper a joint error-control coding and complex-field coding scheme for multi-antenna space-time communications. We rely on OFDM transmitters to convert an FIR MIMO channel to parallel flat fading MIMO channels, which are usually correlated. We adapted the complex-field coding scheme for SISO flat fading channels to space-time flat fading MIMO channels. Then we proposed to serially concatenate error-control coding and complex-field coding with an interleaver in between. The final coded symbols were either transmitted from multiple antennas using antenna-switching, or, using a high-rate triangular signaling scheme. Other ways of transmitting the jointly coded symbols are also possible, including space-time orthogonal designs, delay diversity, and antenna phase sweeping.

We established that the diversity achieved by the joint system with transmit antenna-switching is the product of the free distance of the code, the complex-field encoder size, and the number of receive antennas. The number of transmit antennas does not affect the diversity gain of an antenna-switching scheme, with perfect channel interleaving.

The theoretical results were verified using numerical simulations. It was observed that Hadamard CFC can also offer multiplicative diversity although alone it does not offer full diversity because it is not maximum Hamming distance separable. The coding gain achieved by an error-control coded and Hadamard coded scheme is slightly less than that of an optimum CFC encoder that was previously obtained using algebraic number theory. We have also simulated the proposed scheme in a HIPERLAN setup, and showed that it can significantly improve the convolutionally coded OFDM transmission in the standard, for both single-antenna and multiple-antenna scenarios.

Acknowledgment: This work was supported by the NSF Wireless Initiative Grant No. 99-79443, the NSF Grant No. 01-0516, and by the ARL/CTA Grant No. DAAD19-01-2-011.

Manuscript received on . . .

References


